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Bounds for Solutions of a Six-Point Partial-Difference Scheme

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Abstract—This paper is concerned with a class of partial-difference equations which arise from discretising the one-dimensional heat and the first-order wave equations. Explicit bounds are found for their solutions and subexponential solutions are considered. Using the Green's function, we derive a stability criteria for arbitrary solutions of this equation. © 2004 Elsevier Ltd. All rights reserved.

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1. INTRODUCTION

Before raising objectives of this paper, we shall consider two examples. First, consider an initial value problem involving a hyperbolic equation

$$\begin{aligned} u_t + a u_x &= 0, \qquad && t > 0, \\ u(x,0) &= f(x), && -\infty < x < \infty. \end{aligned}$$

By means of forward in time, forward in space finite difference method for calculating an approximate solution $u(m\Delta x, n\Delta t) = u_m^n$ of this problem, we are led to a difference scheme of the form

$$u_m^{n+1} = (1 + sa)u_m^n - sau_{m+1}^n, \tag{1}$$

where $s = (\Delta t)/(\Delta x)$.

Next, we consider an initial value problem involving the heat equation [1]

$$\begin{aligned} u_t &= \alpha u_{xx}, \qquad && t > 0, \\ u(x,0) &= g(x), && -\infty < x < \infty. \end{aligned}$$

By means of the standard finite difference method [2, p. 165] for calculating an approximate solution $u(m\Delta x, n\Delta t) = u_m^n$ for this problem we can use the following difference scheme:

$$u_m^{n+1} = \left\{ 1 + r\delta_x^2 + \frac{1}{2}r\left(r - \frac{1}{6}\right)\delta_x^4 + \frac{1}{6}r\left(r^2 - \frac{1}{2}r + \frac{1}{15}\right)\delta_x^6 + \cdots \right\} u_m^n,$$

where $r = \alpha(\Delta t)/(\Delta x)^2$ and $\delta_x u_m^n = u_{m+1/2}^n - u_{m-1/2}^n$.

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If only second-order central differences are retained, then we have a classical explicit equation in the following form:

$$u_m^{n+1} = ru_{m-1}^n + (1-2r)u_m^n + ru_{m+1}^n. \tag{2}$$

If second- and fourth-order differences are retained, then we have an explicit equation of the form

$$u_m^{n+1} = \frac{1}{2}\left(2-5r+6r^2\right)u_m^n + \frac{2}{3}r(2-3r)\left(u_{m+1}^n + u_{m-1}^n\right) - \frac{1}{12}r(1-6r)\left(u_{m+2}^n + u_{m-2}^n\right). \tag{3}$$

We know that equation (2) is stable for $r \le 1/2$ and equation (3) is stable for $r \le 2/3$ (see [2]).

In this paper, we will consider the following partial-difference equation which is more general than equations (1)–(3):

$$u_m^{n+1} = au_{m-2}^n + bu_{m-1}^n + cu_m^n + du_{m+1}^n + eu_{m+2}^n, \qquad (m,n) \in W, \tag{4}$$

where $W = \{(m,n);\ m \in Z,\ n = 0,1,2,\dots\}$ and a, b, c, d, and e are real numbers.

A solution of (4) is a real double sequence $u = \{u(m,n)\}_{(m,n)\in W}$ which satisfies (4). We show that under initial condition

$$u_m^0 = \phi_m, \qquad m \in Z, \tag{5}$$

equation (4) has a unique solution.

It is well known that beside the question of existence of solutions, stability behaviour of solutions of difference schemes such as the one described above is also of fundamental importance because this behaviour is related to the question of growth of numerical errors.

2. BOUNDS FOR SOLUTIONS OF EQUATION (4)

In this section, we will derive an explicit bound for solutions of equation (4), in terms of the initial value (5). First of all, we need a precise method for calculating a solution of equation (4).

THEOREM 1. *The partial-difference equation (4) with initial value (5) has a unique solution.*

PROOF. The set W (as defined in Section 1) can be partitioned into equivalence classes $W_0, W_1, \dots$ defined by

$$W_0 = \{(2,0),(1,0),(0,0),(-1,0),(-2,0),(0,1)\},$$
$$W_k = W_k^+ \cup W_k^- \cup \{(0,k+1)\}, \qquad k = 1,2,\dots,$$

where

$$W_k^+ = \bigcup_{l=0}^{k}\{(2k+2-2l,l),(2k+1-2l,l)\}, \qquad k = 1,2,\dots, \tag{6}$$

$$W_k^- = \bigcup_{l=0}^{k}\{(-2k-2+2l,l),(-2k-1+2l,l)\}, \qquad k = 1,2,\dots. \tag{7}$$

For simplicity, we denote $\{(2k+2-2l,l),(2k+1-2l,l)\}$ as the one-stair of W_k^+.

It is clear that $u(W_0)$ can be found uniquely. Indeed, we can see easily that the same is true for $u(W_1)$. Now, we use induction. Assume that for $(i,j) \in W_l$ where $1 \le l \le k$, the solution of (4),(5) is recognized uniquely. Suppose $(i,j) \in W_k^+$. Then, the zero-stair of W_k^+ is the set $\{(2k+2,0),(2k+1,0)\}$, and it is clear that for these points, the solution can be found directly. Now, by induction suppose for up to t-stair of W_k^+, where $0 \le t < k$, the solution of (4),(5) is recognized uniquely. Since the t-stair of W_k^+ is $\{(2k+2-2t,t),(2k+1-2t,t)\}$, then for these points in view of (4) we have

$$u_{2k+2-2t}^t = au_{2k-2t}^{t-1} + bu_{2k+1-2t}^{t-1} + cu_{2k+2-2t}^{t-1} + du_{2k+3-2t}^{t-1} + eu_{2k+4-2t}^{t-1}. \tag{8}$$

Note that by (6) we have $W_{k-1}^{+} = \bigcup_{l=0}^{k-1}\{(2k-2l,l),(2k-1-2l,l)\}$. Thus, the $(t-1)$-stair of W_{k-1}^{+} is $\{(2k-2t+2,t-1),(2k+1-2t,t-1)\}$, Therefore, $u_{2k-2t+1}^{t-1}$ and $u_{2k-2t+2}^{t-1}$ are recognized uniquely by induction. Similarly, $(2k-2t,t-1)$ is in the $(t-1)$-stair of W_{k-2}^{+} and we can see that $\{(2k-2t+3,t-1),(2k+4-2t,t-1)\}$ is the $(t-1)$-stair of W_k^{+}. Thus, by induction and in view of (8), $u_{2k+2-2t}^{t}$ can be found uniquely. Similarly, we see that $u_{2k+1-2t}^{t}$ is recognized.

If $t=k$, then the k-stair of W_k^{+} is $\{(2,k),(1,k)\}$, and by (4) we have

$$u_2^k = au_0^{k-1} + bu_1^{k-1} + cu_2^{k-1} + du_3^{k-1} + eu_4^{k-1}.$$

First, note that $(0,k-1)\in W_{k-2}$. Indeed $(1,k-1)$, and $(2,k-1)$ are in the $(k-1)$-stair of W_{k-1}^{+} and also $(3,k-1)$, $(4,k-1)$ are in the $(k-1)$-stair of W_k^{+}. Thus, by using induction, u_2^k can be found uniquely. Similarly, we see that u_1^k is recognized. Therefore, the solutions of (4),(5) for $(i,j)\in W_k^{+}$ are recognized uniquely. With a similar argument, we can show that the solution of (4),(5) for $(i,j)\in W_k^{-}$ exists and is unique. Finally, for point $(0,k+1)$ we have

$$u_0^{k+1} = au_{-2}^k + bu_{-1}^k + cu_0^k + du_1^k + eu_2^k.$$

We should note that $(0,k)\in W_{k-1}$, and $\{(-2,k),(-1,k)\}\in W_k^{-}$ and $\{(2,k),(1,k)\}\in W_k^{+}$. Thus, u_0^{k+1} is recognized uniquely. This completes the proof. ∎

It is easy to see that u_0^n is determined by the initial values

$$\{\phi_{-2n},\phi_{-2n+1},\dots,\phi_{2n-1},\phi_{2n}\},$$

and therefore, u_m^n is determined by $\{\phi_{m-2n},\dots,\phi_{m+2n}\}$. In the rest of this article, we will use the following notation:

$$\|\phi\|_{m,n} = \max\{|\phi_k|,\ m-2n\le k\le m+2n\}. \tag{9}$$

By using this notation, we can find an explicit bound for solutions of (4),(5). In the next section, we use the following lemma to explain a sufficient condition for the stability of solution.

LEMMA 1. *Suppose $u=\{u_m^n\}_{(m,n)\in W}$ is a solution of the initial value problem (4),(5), then there is a positive number ξ, such that*

$$|u_m^n| \le \xi^n \|\phi\|_{m,n}, \qquad (m,n)\in W. \tag{10}$$

PROOF. Let $\xi=|a|+|b|+|c|+|d|+|e|$. First of all, when $n=0$, in view of (5) we have

$$\left|u_m^0\right| = |\phi_m| \le \|\phi\|_{m,0}, \qquad m\in W.$$

Thus, our assertion holds for $n=0$. The case, $n=1$, is also easy. Now, assume that (10) holds for all $0\le k<n$. Then, for $k=n$, by (4) we have

$$\begin{aligned}|u_m^n| &\le |a|\,\left|u_{m-2}^{n-1}\right| + |b|\,\left|u_{m-1}^{n-1}\right| + |c|\,\left|u_m^{n-1}\right| + |d|\,\left|u_{m+1}^{n-1}\right| + |e|\,\left|u_{m+2}^{n-1}\right| \\ &\le \xi^{n-1}\{|a|\,\|\phi\|_{m-2,n-1} + |b|\,\|\phi\|_{m-1,n-1} + |c|\,\|\phi\|_{m,n-1} \\ &\quad + |d|\,\|\phi\|_{m+1,n-1} + |e|\,\|\phi\|_{m+2,n-1}\}.\end{aligned}$$

It is clear from (9) that

$$\|\phi\|_{i,n-1} \le \|\phi\|_{m,n}, \qquad i=m-2,\dots,m+2.$$

Therefore,

$$|u_m^n| \le \xi^{n-1}\|\phi\|_{m,n}\{|a|+|b|+|c|+|d|+|e|\} = \xi^n\|\phi\|_{m,n}.$$

Now, the proof is complete. ∎

We note that the sequence $\{x(i)\}_{i=-\infty}^{\infty}$ is subexponential if

$$|x(i)| \leq M\alpha^i, \qquad M \geq 0, \quad \alpha \geq 0.$$

Similarly, a double sequence $u = \{u(i,j)\}_{(i,j)\in W}$ is said to be subexponential if

$$|u(i,j)| \leq M\alpha^i\beta^j, \qquad M, \alpha, \beta \geq 0, \quad (i,j) \in W.$$

Subexponential solutions of a partial-difference equation are important since small errors introduced in the initial data may accumulate only in an exponential fashion when the solution of the partial-difference equation is calculated in a recursive manner. Another important aspect is that their Z-transforms exist [3]. It is well known that every solution of a linear difference equation

$$x_{n+k} + p_1 x_{n+k-1} + \cdots + p_k x_n = 0, \qquad n = 0, 1, \ldots,$$

with constant coefficients $p_1, p_2, \ldots, p_k$, is subexponential [4].

Note that solutions of partial-difference equation are not necessarily subexponential. Two counterexamples are given in [3]. In the next theorem, we prove that if the initial value (5) is subexponential, then the solution of (4),(5) is subexponential.

THEOREM 2. *Suppose $\phi = \{\phi_m\}_{m\in Z}$ is subexponential such that*

$$|\phi_m| \leq M_1\alpha^m, \qquad m \in Z, \quad M_1, \alpha \geq 0.$$

Then, the solution of (4) subject to initial value (5) is subexponential.

PROOF. Letting $M = M_1$, $\lambda = \alpha$, $\eta = (|a|/\alpha^2) + (|b|/\alpha) + |c| + |d|\alpha + |e|\alpha^2$, we assert that

$$|u_m^n| \leq M\lambda^m\eta^n, \qquad (m,n) \in W. \tag{11}$$

Note that $|u_m^0| = |\phi_m| \leq M\lambda^m$. Now, we use induction. Suppose that (11) holds for all $0 \leq j < n$. For $j = n$, by (4) we have

$$\begin{aligned} |u_m^n| &\leq |a| \left|u_{m-2}^{n-1}\right| + |b| \left|u_{m-1}^{n-1}\right| + |c| \left|u_m^{n-1}\right| + |d| \left|u_{m+1}^{n-1}\right| + |e| \left|u_{m+2}^{n-1}\right| \\ &\leq M \left\{|a|\lambda^{m-2}\eta^{n-1} + |b|\lambda^{m-1}\eta^{n-1} + |c|\lambda^m\eta^{n-1} + |d|\lambda^{m+1}\eta^{n-1} + |e|\lambda^{m+2}\eta^{n-1}\right\} \\ &\leq M\lambda^m\eta^{n-1} \left\{\frac{|a|}{\lambda^2} + \frac{|b|}{\lambda} + |c| + |d|\lambda + |e|\lambda^2\right\} = M\lambda^m\eta^n. \end{aligned}$$

This completes the proof. ∎

3. GREEN'S FUNCTION AND STABILITY

A solution of equation (4) will be called the Green's function of (4) if it satisfies the initial condition

$$u_m^0 = \delta_{m0}, \qquad m \in Z,$$

where δ_{ij} is the Kronecker delta function. From Theorem 1, we know that such a solution exists and is unique. This solution will be denoted by $G = \{G_m^n\}$. When $a = e = 0$, G_m^n is given by the coefficient of the term x^m, in the expansion of $(bx + c + dx^{-1})^n$, for $|m| \leq n$, see [5]. In this section, we extend this result for equation (4).

THEOREM 3. *The $(m,n)^{th}$ component G_m^n, of the Green's function $G = \{G_m^n\}$, of (4) is the coefficient of the term x^m, in the expansion of rational function $(ax^2 + bx + c + dx^{-1} + ex^{-2})^n$, for $|m| \le 2n$ and is zero otherwise.*

Before we prove this theorem, note that the primary role of x is being an ordering parameter since the coefficients of powers of x which are significant. Thus, we are not interested in the case $x = 0$.

PROOF. It is clear that when $n = 0$, the function $(ax^2 + bx + c + dx^{-1} + ex^{-2})^n = 1$, and the 0^{th} horizontal vector of G is $\{\dots, 0, 1, 0, \dots\}$, and thus our assertion holds when $n = 0$. The case $n = 1$ is easy. Assume that the case $n = k$ holds, that is

$$\left(ax^2 + bx + c + dx^{-1} + ex^{-2}\right)^k = \sum_{i=-2k}^{2k} G_i^k x^i.$$

Then, for $n = k + 1$, we have

$$\begin{aligned}
&\left(ax^2 + bx + c + dx^{-1} + ex^{-2}\right)^{k+1} \\
&= \left(\sum_{i=-2k}^{2k} G_i^k x^i\right)\left(ax^2 + bx + c + dx^{-1} + ex^{-2}\right) \\
&= eG_{-2k}^k x^{-2k-2} + \left\{eG_{-2k+1}^k + dG_{-2k}^k\right\} x^{-2k-1} + \left\{eG_{-2k}^k + dG_{-2k+1}^k + cG_{-2k}^k\right\} x^{-2k} + \cdots \\
&\quad + \left\{bG_{2k}^k + aG_{2k-1}^k\right\} x^{2k+1} + aG_{2k}^k x^{2k+2} \\
&= \left\{aG_{-2k-4}^k + bG_{-2k-3}^k + cG_{-2k-2}^k + dG_{-2k-1}^k + eG_{-2k}^k\right\} x^{-2k-2} + \cdots \\
&\quad + \left\{aG_{i-2}^k + bG_{i-1}^k + cG_i^k + dG_{i+1}^k + eG_{i+2}^k\right\} x^i + \cdots \\
&\quad + \left\{aG_{2k}^k + bG_{2k+1}^k + cG_{2k+2}^k + dG_{2k+3}^k + eG_{2k+4}^k\right\} x^{2k+2} = \sum_{i=-2k-2}^{2k+2} G_i^{k+1} x^i.
\end{aligned}$$

Now, by using induction the proof is complete. ∎

We note that the convolution of two sequences $\phi = \{\phi_m\}_{m\in Z}$, and $\psi = \{\psi_m\}_{m\in Z}$, is the sequence $\phi * \psi$, defined by

$$(\phi * \psi)_m = \sum_{k=-\infty}^{\infty} \phi_k \psi_{m-k}, \qquad m \in Z.$$

Whenever one of the sequences ϕ and ψ has a finite number of nonzero terms, such a sequence exists. By means of Green's function G, we will be able to show that the solution of (4),(5) can be expanded as a convolution.

THEOREM 4. *The solution $u = \{u_m^n\}$, of the initial value problem (4),(5) is given by*

$$u_m^n = \sum_{k=-\infty}^{\infty} G_k^n \phi_{m-k} = \sum_{k=-2n}^{2n} G_k^n \phi_{m-k}, \qquad (m,n) \in W.$$

PROOF. The proof is done by direct verification. When $n = 0$, we have

$$\sum_{k=-\infty}^{\infty} G_k^0 \phi_{m-k} = \sum_{k=-\infty}^{\infty} \delta_{k0} \phi_{m-k} = \phi_m = u_m^0, \qquad m \in Z.$$

Assume by induction that the case $n = j$ holds. For $n = j+1$, by (4) we have

$$\begin{aligned} u_m^{j+1} &= a\left(\sum_{k=-\infty}^{\infty} G_k^j \phi_{m-2-k}\right) + b\left(\sum_{k=-\infty}^{\infty} G_k^j \phi_{m-1-k}\right) \\ &\quad + c\left(\sum_{k=-\infty}^{\infty} G_k^j \phi_{m-k}\right) + d\left(\sum_{k=-\infty}^{\infty} G_k^j \phi_{m+1+k}\right) + e\left(\sum_{k=-\infty}^{\infty} G_k^j \phi_{m+2-k}\right) \\ &= \sum_{k=-\infty}^{\infty} \left\{aG_{k-2}^j + bG_{k-1}^j + cG_k^j + dG_{k+1}^j + eG_{k+2}^j\right\} \phi_{m-k} = \sum_{k=-\infty}^{\infty} G_k^{j+1} \phi_{m-k}. \end{aligned}$$

It is easy to see that

$$G_m^j = 0, \qquad |m| > 2j.$$

Thus, we have no problem with the convergence of the series and the second equality holds. So, the proof is complete. ∎

There are several immediate consequences of Theorems 3 and 4. For instance if $\phi = \{1\}_{m\in Z}$, then the solution of (4),(5) is given by

$$u_m^n = \sum_{k=-2n}^{2n} G_k^n = (a+b+c+d+e)^n. \tag{12}$$

Also, if $\phi = \{(-1)^m\}_{m\in Z}$, then the solution of (4),(5) is given by

$$u_m^n = \sum_{k=-2n}^{2n} G_k^n (-1)^k = (a-b+c-d+e)^n. \tag{13}$$

Now, we explain one of the properties of Green's function of equation (4).

PROPOSITION 1. *If $a+b+c+d+e < 0$, then there exist integers m_1 and m_2 such that*

$$G_{m_1}^n G_{m_2}^{n+1} < 0, \qquad n = 0, 1, 2, \dots.$$

PROOF. Since $(a+b+c+d+e)^n (a+b+c+d+e)^{n+1} < 0$ for $n \ge 0$, then by equation (12) we have

$$\left\{\sum_{k=-2n}^{2n} G_k^n\right\} \left\{\sum_{k=-2n-2}^{2n+2} G_k^{n+1}\right\} < 0.$$

This implies our assertion. ∎

Proposition 1 shows that when $a+b+c+d+e < 0$, then adjacent horizontal vectors of the Green's function must have components with different signs.

We note that the solution of equation (4) is said to be stable if there exists a constant Γ such that for every initial sequence $\phi = \{\phi_m\}_{m=-\infty}^{\infty}$ the corresponding solution of (4),(5) satisfies

$$|u_m^n| \le \Gamma \|\phi\|_{m,n}, \qquad (m,n) \in W.$$

See [5] for details.

Now, we derive stability criterion for the solutions of the initial value problem (4),(5).

THEOREM 5. *Let $u = \{u_m^n\}$ be a solution of the initial value problem (4),(5). If $|a| + |b| + |c| + |d| + |e| \le 1$, then the solution is stable.*

PROOF. By Lemma 1 we have

$$|u_m^n| \le (|a| + |b| + |c| + |d| + |e|)^n \|\phi\|_{m,n}, \qquad (m,n) \in W.$$

Thus, $|u_m^n| \le \|\phi\|_{m,n}$. Therefore, by the definition of stability, the proof is complete. ∎

The converse of Theorem 5 is not true. For example, if we use this theorem for equation (3) we cannot find the stability criterion. In the next theorem, we will find some conditions under which the converse of Theorem 5 is true.

THEOREM 6. *If $a, b, c, d, e \geq 0$, or $a, c, e > 0$, and $b, d < 0$, then the condition $|a| + |b| + |c| + |d| + |e| \leq 1$ is necessary for stability of solutions of initial value problem (4).*

PROOF. Suppose $a, b, c, d, e \geq 0$, and $|a| + |b| + |c| + |d| + |e| > 1$. Let $\phi = \{p\}_{m \in Z}$, where $p > 0$ is a constant. By Theorem 4 and equation (12) we have

$$u_m^n = \sum_{k=-2n}^{2n} G_k^n = p(a + b + c + d + e)^n \|\Phi\|_{m,n}.$$

Since $a+b+c+d+e > 1$, then for any $\Gamma > 0$ there exists an integer n such that $(a+b+c+d+e)^n > \Gamma$. Thus, $|u_m^n| > \Gamma\|\phi\|_{m,n}$, which shows that the trivial solution of equation (4) is not stable.

Again, suppose $a, c, e > 0$ and $d, b < 0$. Let $\phi = \{(-1)^m\}_{m \in Z}$. Then, by equation (13) and Theorem 4 we have

$$u_m^n = \sum_{k=-2n}^{2n} G_k^n (-1)^{m-k} = (-1)^m \sum_{k=-2n}^{2n} G_k^n (-1)^k = (-1)^m (e - d + c - b + a)^n.$$

Now, suppose $|a| + |b| + |c| + |d| + |e| > 1$. Since $a - d + c - b + a > 1$, then for any $\Gamma > 0$, there exists an integer n, such that $(a - d + c - b + a)^n > \Gamma$. Thus, $|u_m^n| > \Gamma\|\phi\|_{m,n}$, which shows that the trivial solution of equation (4) is not stable. This completes the proof. ∎

As an example consider equation (4), with $a = e = 0$ and $b = d = r$, and $c = 1 - 2r$. The condition $|a| + |b| + |c| + |d| + |e| \leq 1$ is equivalent to $r \leq 1/2$, which shows that the trivial solution of equation (2) for $r \leq 1/2$ is stable, as we mentioned in Section 1.

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